EXISTENCE OF A UTILITY IN INFINITE DIMENSIONAL PARTIALLY ORDERED SPACES*

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ABSTRACT

An example is given of a preference order on a space of denumerable algebraic dimension that has no utility, and necessary and sufficient conditions for the existence of utilities in various linear spaces are given.

1. **Introduction.** "Utility" is a numerical function representing a preference order, which is useful in solving optimization problems. Until recently, utility theory was restricted to complete preference orders; the extension to orders that are not necessarily complete was made in [1], but the results of that paper are restricted to finite-dimensional spaces of alternatives. The infinite-dimensional case is important for certain types of economic models, e.g. those involving an infinite time horizon or continuous time, commodities that may be available in a continuum of quantities, continua of prices or of geographic locations, etc. It is the object of this paper to investigate to what extent the results of [1] can be extended to infinite-dimensional spaces of alternatives. In particular, a problem raised in [1] will be answered.

Let X be a real linear space. We assume that on X there is defined a transitive and reflexive relation called preference-or-indifference and denoted by $\succsim$. If $x \succsim y$ and $y \succsim x$, we shall say that x is indifferent to y and write $x \sim y$. If $x \succsim y$ but not $x \sim y$, we shall say that x is preferred to y and write $x \succ y$. The relation $\succsim$ will be called a partial order. (We shall not assume that $\succsim$ is complete).

We assume that the following conditions hold:

(1.1) $x \succsim y$ implies $x + z \succsim y + z$ for all $z \in X$;

(1.2) $x \succsim y$ and $\alpha > 0$ implies $\alpha x \succsim \alpha y$;

(1.3) $x \succ kz$ for all positive integers k implies not $z \succ 0$.

A real linear functional u defined on all X will be called a utility if $x \succsim y$ implies $u(x) \geqq u(y)$, and $x \succ y$ implies $u(x) > u(y)$. A vector valued linear function v defined on all X will be called a multi-dimensional utility if $x \succsim y$ implies $v(x) \succsim v(y)$ and $x \succ y$ implies $v(x) \succ v(y)$. Here the order on the vector space

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(the space of values of v) is the lexicographic order, i.e., the vector $v = (v_1, \dots, v_n)$ is preferred or indifferent to $w = (w_1, \dots, w_n)$ if $v = w$ or if $v_i > w_i$ for the first coordinate i such that $v_i \neq w_i$. (The lexicographic order satisfies (1.1), (1.2) but not (1.3)).

Aumann [1] proved that if X is a finite-dimensional Euclidean space, the assumptions (1.1)–(1.3) imply the existence of a utility. He gave an example of a partial order satisfying these assumptions without having any utility, if X is the set of all infinite sequences of real numbers, and raised the problem of the existence of a utility if X has countable dimensionality (algebraic dimensionality).

An example will be given here of an order on the space with countable dimensionality which does not have even a multi-dimensional (finite-dimensional) utility and, of course, does not have a (numerical) utility. A necessary and sufficient condition for the existence of a utility in the countable dimensional case will be given, and a sufficient condition for the existence of a utility if X is a seperable normed space will also be given.

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2. **Some preliminaries.** The following are simple consequences of the assumptions (1.1)–(1.3). (We denote the zero element of X by 0):

(2.1) $x \succsim y$ if and only if $x - y \succsim 0$;
$x \succ y$ if and only if $x - y \succ 0$;

(2.2) $x \succsim y$ and $z \succsim y$ and $0 < \alpha < 1$ imply $x + (1-\alpha)z \succsim y$;
$x \succ y$ and $z \succ y$ and $0 < \alpha < 1$ imply $x + (1-\alpha)z \succ y$;

(2.3) $x \succsim 0$ if and only if $-x \precsim 0$, $x \succ 0$ if and only if $-x \prec 0$.

Set $T = \{x : x \succ 0\}$, $S = \{x : x \succsim 0\}$.

(2.4) T and S are convex cones, $0 \notin T$, $0 \in S$ and $T \subset S$.
If $A, B \subset X$, set $A + B = \{a + b; a \in A, b \in B\}$, $-A = \{x : -x \in A\}$

(2.5) $\{x : x \sim 0\} = S \cap (-S)$ and is a linear subspace of X.

We remark that a real linear functional u defined on X so that $u(x) \geqq 0$ if $x \in S$, $u(x) > 0$ if $x \in T$, is a utility. (It is obvious that a utility satisfies this). Similarly, a vector valued linear function v defined on X is a multi-dimensional utility if and only if $x \in S$ implies $v(x) \succsim 0$, $x \in T$ implies $v(x) \succ 0$ (preference in the lexicographic order).

THEOREM A. *A necessary condition for the existence of a utility for the partial order $\succsim$ is that in every linear topology on X in which every linear functional defined on X is continuous,*

(2.6) $(-T) \cap \bar{S} = \varnothing$ (which implies of course that $(-T) \cap \bar{T} = \varnothing$).*

* A superior bar denotes closure.

Proof. Assume, on the contrary, that there is a point $x \in (-T) \cap \bar{S}$. From (2.3) it follows that $u(x) < 0$. In every neighbourhood $U(x)$ of x there is a point y with $y \in S$ and so $u(y) \geqq 0$. Hence by the continuity of u, $u(x) \geqq 0$, a contradiction.

This theorem shows that condition (1.3) is necessary for the existence of a utility in R^n. Otherwise there are x, y, z with $x \succ kz$, $z \prec 0$. Hence $(x/k) \succ z$ and by (2.1), $x/k - z \succ 0$. But in the usual topology on R^n $(x/k) - z \to -z$, so that $-z \in \bar{S}$, contradicting (2.6). (Every linear functional defined on R^n is continuous in the usual topology).

In R^n (1.3) implies (2.6) ([1, page 456]).

A partial order is called pure if $x \sim y$ implies $x = y$. A partial order is pure if and only if $x \sim 0$ implies $x = 0$, or equivalently, if and only if $S = T \cup \{0\}$.

3. **An example.** Let X be the space of the sequences of real numbers which have only a finite number of members different from zero. X has denumerable dimensionality and up to isomorphism is the only such space. The algebraic dual space of X is the space of all real sequences. Developing an earlier example of mine, M. Perles gave the following example of a pure partial order on X which satisfies (1.1)–(1.3) and has no utility. In fact, this order has no finite-dimensional utility.

Denote by e_i the i-th unit vector. Let $e_{n,1} = e_1$, $e_{n,i} = -(n-1)e_{i-1} + e_i$, $i = 2, \ldots, n$. ($e_{n,i}$ is the vector of X whose $(i-1)$-th coordinate is $-(n-1)$ and whose i-th coordinate is 1, all the other coordinates being zero). For example:

$$e_{4,1} = (1,0,\ldots,0,\ldots), e_{4,2} = (-3,1,0,\ldots), e_{4,3} = (0,-3,1,0\ldots),$$
$$e_{4,4} = (0,0,-3,1,0,\ldots)$$

Define P_n as the set of all linear combinations of the form $\sum_{i=1}^{n} \alpha_i e_{n,i}$ with $\alpha_i \geqq 0$, $\alpha_n > 0$. We shall show presently that $\bigcup_{n=1}^{\infty} P_n$ and $\bigcup_{n=1}^{N} P_n$ are convex cones in X. It is obvious that each P_n is a convex cone. $e_n \in P_n$, since $e_i = (n-1)^{i-1} e_{n,1} + (n-1)^{i-2} e_{n,2} + \ldots + (n-1) e_{n,i-1} + e_{n,i}$. Let $x \in P_n, y \in P_m$ and assume $n > m$. Then

$$x = \sum_{i=1}^{n} \alpha_i e_{n,i}, \qquad y = \sum_{i=1}^{m} \beta_i e_{m,i} \qquad \text{with } \alpha_n, \beta_m > 0,\ \alpha_i, \beta_i \geqq 0.$$

$$\begin{aligned} x + y &= \sum_{i=1}^{m} (\alpha_i + \beta_i) e_{n,i} + (n-m) \sum_{i=2}^{m} \beta_i e_{i-1} + \sum_{i=m+1}^{n} \alpha_i e_{n,i} \\ &= \sum_{i=1}^{m} (\alpha_i + \beta_i) e_{n,i} + (n-m) \sum_{i=2}^{m} \beta_i \left(\sum_{k=1}^{n-1} (n-1)^{i-k-1} e_{n,k} \right) \\ &\quad + \sum_{i=m+1}^{n} \alpha_i e_{n,i} \end{aligned}$$

and so $x + y$ is contained in P_n.

Define $T = \bigcup_{n=1}^{\infty} P_n$, and set $x \succ y$ if and only if $x - y \in T$. We obtain a pure order satisfying (1.1)–(1.3), $(0 \notin T)$. It is obvious that (1.1) and (1.2) are satisfied. Set $E^n = \{x \in X; x_i = 0 \text{ for } i > n\}$. It suffices to prove (1.3) for $x, z \in E^n$, $n = 1, 2, \ldots$. By theorem A, it suffices to show that the partial order, reduced to E^n, has a utility. Define $u_n \in E^{n*}$ by

$$u_n(x) = \sum_{k=1}^{n} n^k x_k.$$

From the definitions of T and the P_i's it follows that $x \in T \cap E^n$ if and only if $x \in P_i$, $i = 1, \ldots, n$. Now $u_n(e_{i,j}) > 0$ for $1 \leqq i \leqq n$, $1 \leqq j \leqq i$. Hence $u_n(x) > 0$ for $x \in T \cap E^n$ and therefore u_n is a utility on E^n and our order satisfies (1.3).

T has no finite-dimensional utility. Suppose that $v(x)$ is such a utility, $v(x) = (\phi_1(x), \ldots, \phi_m(x))$, $\phi_i(x)$ are linear functionals on X, and without loss of generality let m be the minimal dimension of a possible multi-dimensional utility for this order (this includes the case $m = 1$). Then ϕ_1 is not identically zero. Every unit vector e_i is contained in T. Hence there is an e_k with $\phi_1(e_k) > 0$ so that $\phi_1(-e_k) < 0$. Let n be an integer, $n > k + 2$. For every $\varepsilon > 0$, the vector

$$a_{k,n,\varepsilon} = -e_k + \frac{1}{n-1} e_{k+1} - \varepsilon(n-1)e_{n-1} + \varepsilon e_n$$

$$= \Big(0, \ldots, 0, \underset{k}{-1}, \underset{k+1}{\frac{1}{n-1}}, \ldots, \underset{n-1}{-\varepsilon(n-1)}, \underset{n}{\varepsilon}, 0, \ldots\Big)$$

is contained in P_n and therefore in T, so that $\phi_1(a_{k,n,\varepsilon}) \geqq 0$, (if $\phi_1(x) < 0$ then 0 is preferred to $v(x)$ in the lexicographic order). By letting $\varepsilon \to 0$ it follows that $\phi_1(-e_k + (1/(n-1))e_{k+1}) \geqq 0$, and by letting $n \to \infty$ it follows that $\phi_1(-e_k) \geqq 0$, a contradiction. (It is clear that every linear functional on X, reduced to E^n, is continuous on E^n).

4. **The main existence theorems.** Let X be the space of the real sequences which have only a finite number of members different from zero, $\succsim$ a partial order on X satisfying (1.1)–(1.2). We may assume that $\succsim$ is pure. Otherwise divide by $E = \{x : x \sim 0\}$ which is a linear subspace of X. On the quotient space X/E, $\succsim$ induces a pure partial order satisfying (1.1)–(1.2). The quotient space is isomorphic to a linear subspace of X, which is either a Euclidean space or has denumerable dimensionality, i.e. is isomorphic to X. The first case is settled in [1], and it follows that (1.3) is a necessary and sufficient condition for the existence of a utility. (It is obvious that a utility defined on X/E induces in a natural way a utility defined on X.)

In order to settle the second case, let us topologize X in the following way: a typical neighbourhood of zero is the set of all $x \in X$ such that $|x_i| < \varepsilon_i$ for a given sequence of positive numbers $(\varepsilon_i)_{i=1}^{\infty}$.

Every linear functional defined on X is continuous in this topology. For if $u \in X^*$ (the algebraic dual space of X), then u is represented by a sequence of real numbers u_i where $u_i = u(e_i)$. Let an $\varepsilon > 0$ be given. Define $\varepsilon_i = \varepsilon/2^i u_i$ if $u_i \neq 0$, $\varepsilon_i = 1$ if $u_i = 0$. Let x be a vector in X. For every $y \in X$, $|(y-x)_i| < \varepsilon_i$ implies $|u(y) - u(x)| < \varepsilon$, hence u is continuous. (The topology induced by any one of the l_p norms does not have this property).

We remark that this topology is separable, since in every E^m there is a dense sequence, and the union of these sequences is dense in X. Moreover, the induced topology on E^n has a countable basis for every n (the induced topology coincides with the usual topology). Hence, if $\mathscr{M} \subset X$ and to each $x \in \mathscr{M}$ there corresponds an open set U_x which contains x, then there is a sequence $\{x_i\}$ of points of $\mathscr{M}$ such that $\mathscr{M} \subset \bigcup_i U_{x_i}$. We may construct this sequence by first covering $\mathscr{M} \cap E^n$ and then taking the union of these sequences (union over n).

We are now able to state and prove the following theorem:

THEOREM B. *Let* $\succsim$ *be a pure order on* X *and let* $(-T) \cap \bar{T} = \varnothing$ *in the above topology. Then there is a utility on* X.

Proof. Let p be any point of T. Following Klee ([3]), we assert that there is a neighbourhood U_p of p such that $[U_p \cup T]$ (the convex hull of U_p and T) does not contain 0. Otherwise there are $q \in U_p$, $a \in T$, $0 < \alpha < 1$ such that $\alpha q + (1-\alpha)a = 0$, for every U_p. Then $-\alpha q = (1-\alpha)a$, $-q = (1-\alpha)/\alpha)a \in T$ (since T is a cone) so that $-p \in \bar{T}$, contradicting $(-T) \cap \bar{T} = \varnothing$.

$[U_p \cup T]$ is a convex set with non-empty interior, and $0 \notin [U_p \cup T]$. Hence there is a nonzero linear functional u_p which supports it, i.e., $x \in [U_p \cup T]$ implies $u_p(x) \geqq 0$ ([4 page 191]). Since p is an internal point of $[U_p \cup T]$, $u_p(p) > 0$. There exists a neighborhood V_p of p with $u_p(y) > 0$ for $y \in V_p$, because u_p is continuous. By the remark made above, there is a sequence p_i of points of T such that $T \subset \bigcup_i V_{p_i}$, since $T \subset \bigcup_p V_p$.

$u_{p_n}|E^n$ is a linear functional on E^n and so is bounded there. Let us denote its Euclidean norm on E^n by $\|u_{p_n}\|_n$. Set

$$u(x) = \sum_{n=1}^{\infty} \frac{u_{p_n}(x)}{2^n \|u_{p_n}\|_n + 1}.$$

The series converges pointwise for each $x \in X$, since there is a positive integer m with $x \in E^m$, so that $x \in E^i$ for all $i \geqq m$. For every $n \geqq m$,

$$\sum_{i=1}^{n} \left| \frac{u_{p_i}(x)}{2^i \|u_{p_i}\|_i + 1} \right| \leqq \sum_{i=1}^{m-1} \frac{|u_{p_i}(x)|}{2^i \|u_{p_i}\|_i + 1} + \sum_{i=m}^{n} \frac{\|u_{p_i}\|_i \|x\|_i}{2^i \|u_{p_i}\|_i + 1}$$

$$\leqq \sum_{i=1}^{m-1} \frac{|u_{p_i}(x)|}{2^i \|u_{p_i}\|_i + 1} + \sum_{i=m}^{n} \frac{\|x\|}{2^i}$$

$$\left(\|x\|_i = \|x\| = \left(\sum_{j=1}^{m} x_j^2 \right)^{1/2} \text{for } i \geqq m \right).$$

The first term on the right hand of this inequality is a constant for a fixed x, and the second term converges. For every $x \in T$, $u(x) > 0$, since $u_{p_n}(x) \geqq 0$ for all n and there is at least one k with $x \in V_{p_k}$, so that $u_{p_k}(x) > 0$. Thus u is a utility.

From theorems A and B and by the above remarks it follows that (2.6) is both necessary and sufficient in the countable dimensional case.

THEOREM C. *If X is a separable normed linear space, $\succsim$ is a partial order on X satisfying* (1.1), (1.2), (2.6) *then there is a (bounded) utility. (This includes the case of l_p spaces, $1 \leqq p < \infty$.)*

Proof. By a theorem of Klee ([3 theorem (2.7)]) there is a functional u such that $x \in \bar{S}$ implies $u(x) \geqq 0$, and $u(x) > 0$ for $x \in \bar{S}$ and $x \notin -\bar{S}$ (hence $u(x) = 0$ for $x \in \bar{S} \cap (-\bar{S})$).

It is easy to see, by the method of proof of theorem A, that (2.6) is also necessary for the existence of a *bounded* utility.

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